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1983 J. Phys. A: Math. Gen. 16 1611

(http://iopscience.iop.org/0305-4470/16/8/009)

J. Phys. A: Math. Gen. 16 (1983) 1611–1621. Printed in Great Britain

A kink–antikink soliton propagating with varying velocity

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Received 24 August 1982, in final form 27 October 1982

Abstract. Using our previous KdV solution, we apply a Bäcklund transformation to obtain new sets of solutions to another nonlinear equation $y_t + y_{xxx} - 6y^2y_x + 6y_x = 0$ based on theorems deduced by us recently. The solutions to the above equation can be separated into three categories: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$. When $\lambda = 0$, the above equation becomes the modified KdV (MKdV) equation. From these solutions ($\lambda \neq 0$), we have also obtained new solutions (which also contain a non-zero parameter λ) to the standard MKdV equation. In this investigation, we analyse in detail our solutions for the cases $\lambda > 0$ and $\lambda = 0$. Many numerical examples are given to illustrate the main features of some of our new solutions.

1. Introduction

Recently, we have obtained a set of solutions to the KdV equation via the Bäcklund transformation (Au and Fung 1982a, b, Wahlquist and Estabrook 1975) using a differential geometrical approach. In our derivation, we have emphasised the physical meaning of a certain parameter, called the vacuum parameter b, which appears in the solutions. Since the value of the vacuum parameter controls certain physically observable situations (like the direction, magnitude of the velocity and amplitude of the solitary wave), we are led by our deduction to believe that b represents physically observable effects in a KdV soliton. It is natural to try to find out whether solutions of other nonlinear equations contain physically significant vacuum state parameters.

In the second paper of our series (Fung and Au 1982), using the close-ideal condition in our differential geometrical approach which guarantees the integrability condition, we have built a bridge connecting the solutions of the KdV equation and the nonlinear equation (10) of which the MKdV equation is a special case. The theorems we have established in our second paper are rather powerful, covering ground of which the Miura transformation represents part of our area. Using the above Bäcklund transformation techniques, we have obtained a set of kink–antikink solutions to (10), where the vacuum parameter has physical meaning, but is no longer simply b.

To follow up our research work, using a KdV solution $u = -1/x^2$ we employ our Bäcklund transformation again to obtain a new set of solutions to (10) in this paper, using the theorems we have just established. We have discovered that one new set of solutions to (10) has singularity properties and the kink–antikink soliton propagates to the right with a varying velocity in a transient domain of space and time. When the parameter λ is zero in (10), we have the MKdV equation. The new set of MKdV solutions is not a common one-soliton or kink–antikink solution, but has rather interesting propagating properties. In order to demonstrate the main features of our discovery, we have performed numerical analysis on our new solutions in this investigation.

2. Solutions to the nonlinear equation $y_t + y_{xxx} - 6y^2 y_x + 6\lambda y_x = 0$

We have obtained solutions to the KdV equation

$$u_t + u_{xxx} + 12uu_x = 0. \tag{1}$$

Using the Bäcklund transformation

$$u^* = b \tag{2}$$

$$u^* = u(x, t) \tag{3}$$

$$u^* = -u(x, t) - y^2 + \lambda \tag{4}$$

where the function $y(x, t)$ is restricted by the conditions

$$y_x = -2u(x, t) - y^2 + \lambda \tag{5}$$

$$y_t = -4[u(x, t) + \lambda]y_x + 2u_{xx} - 4u_x y \tag{6}$$

these solutions are

$$u^* = b \tag{7}$$

$$u^* = b - (x - 12bt - x_0)^{-2} \qquad \text{for } \lambda = 2b \tag{8}$$

$$u^* = \lambda - b - (\lambda - 2b)\frac{\{C \exp[(\lambda - 2b)^{1/2} r] - \exp[-(\lambda - 2b)^{1/2} r]\}^2}{\{C \exp[(\lambda - 2b)^{1/2} r] + \exp[-(\lambda - 2b)^{1/2} r]\}^2} \tag{9}$$

where $r = x - 4(b + \lambda)t$, and b and C are constants.

We have proved (Fung and Au 1982) that if u satisfies the KdV equation, then the solution to y thus obtained by the Bäcklund transformation (5)–(6) satisfies the nonlinear equation

$$y_t + y_{xxx} - 6y^2 y_x + 6\lambda y_x = 0 \tag{10}$$

and *vice versa*; namely if y satisfies (10), then $u = -\frac{1}{2}(y^2 + y_x - \lambda)$ satisfies the KdV equation. In this investigation, we shall take solution (8) under the special condition $b = 0$ (and $x_0 = 0$), namely

$$u = -1/x^2, \tag{11}$$

and find a new set of solutions to (10). Along these lines we first write (6) in the following form, using (11):

$$4(\lambda - 1/x^2)y_x + y_t = -4(3/x^4 + 2y/x^3). \tag{12}$$

The characteristic equations of (12) are

$$\frac{dx}{4(\lambda - 1/x^2)} = dt = \frac{dy}{-4(3/x^4 + 2y/x^3)}. \tag{13}$$

We can readily solve equations (13) for three different domains of the parameter λ:

$$y=\frac{1}{x(\lambda x^2-1)}+\frac{\lambda x^2}{|\lambda x^2-1|}(\pm\sqrt{\lambda}) \qquad (\text{for } \lambda>0) \tag{14}$$

$$y=\frac{1}{x(\lambda x^2-1)}+\frac{\lambda^{3/2}x^2}{(\lambda x^2-1)}\left(\frac{C\exp(\sqrt{\lambda}\xi)-\exp(-\sqrt{\lambda}\xi)}{C\exp(\sqrt{\lambda}\xi)+\exp(-\sqrt{\lambda}\xi)}\right) \qquad (\text{for } \lambda\neq 0) \tag{15}$$

$$y=-\frac{1}{x}+\frac{x^2}{\frac{1}{3}x^3+4t+D} \qquad (\text{for } \lambda=0) \tag{16}$$

where D is a constant and the time-dependent argument ξ is given by

$$\xi=x+\frac{1}{2\sqrt{\lambda}}\ln\left(\frac{|\sqrt{\lambda}\,x-1|}{|\sqrt{\lambda}\,x+1|}\right)-4\lambda t \qquad (\text{for } \lambda>0) \tag{17}$$

$$\xi=x-\frac{1}{k}\tan^{-1}(kx)+4k^2t \qquad (\text{for } \lambda<0) \tag{18}$$

while $k^2=-\lambda$.

We would remark that for $\lambda<0$, solution (15) remains real. We have substituted solutions (14)–(16) back into (10), to check that they satisfy the new nonlinear equation (10). Note that if $y(x,t)$ satisfies (10), then $-y(x,t)$ also satisfies (10).

It is worth noting that the equation under study (namely (10)) can be transformed to the MKdV equation

$$y_t+y_{xxx}-6y^2y_x=0 \tag{19}$$

via the transformation

$$t'=t \qquad x'=x-6\lambda t. \tag{20}$$

It is easy to verify by direct substitution that if $y(x,t)$ is a solution to (10), then $y(x+6\lambda t,t)$ is a solution to (19). The reverse state is also true: if $y(x,t)$ is a solution to (19), then $y(x-6\lambda t,t)$ is a solution to (10). Based on our investigation presented in this section, while expressions (14) and (15) are solutions to (10), accordingly the following expressions (21) and (22) are respectively the solutions to (19):

$$y=\frac{1}{(x+6\lambda t)[\lambda(x+6\lambda t)^2-1]}+\frac{\lambda(x+6\lambda t)^2}{|\lambda(x+6\lambda t)^2-1|}\pm\sqrt{\lambda} \tag{21}$$

$$y=\frac{1}{(x+6\lambda t)[\lambda(x+6\lambda t)^2-1]}+\frac{\lambda^{3/2}(x+6\lambda t)^2}{[\lambda(x+6\lambda t)^2-1]}\left(\frac{C\exp(\sqrt{\lambda}\eta)-\exp(-\sqrt{\lambda}\eta)}{C\exp(\sqrt{\lambda}\eta)+\exp(-\sqrt{\lambda}\eta)}\right) \tag{22}$$

where

$$\eta=\left[x+\frac{1}{2\sqrt{\lambda}}\ln\left(\frac{|\sqrt{\lambda}(x+6\lambda t)-1|}{|\sqrt{\lambda}(x+6\lambda t)+1|}\right)+2\lambda t\right] \qquad \text{for } \lambda>0$$

$$\eta=x-k^{-1}\tan^{-1}[k(x+6\lambda t)]+2\lambda t \qquad \text{for } \lambda<0$$

$$k^2=-\lambda.$$

In the reverse direction, based on solution (16), it is elementary to find that

$$y = -\frac{1}{(x-6\lambda t)} + \frac{3(x-6\lambda t)^2}{(x-6\lambda t)^3 + 12t + D} \tag{23}$$

is the corresponding solution to (10).

We get new solutions to (10) via our Bäcklund transformation (5), (6) which contains a rather general parameter λ. If we set $\lambda = 0$ at the beginning (namely, attempting to solve MKdV instead of (10)), we would lose some solutions for equation (10) and the MKdV equation. In the example we have undertaken if we set $\lambda = 0$, we would have lost solutions (14), (15) (to equation (10)) and the corresponding solutions (21), (22) (to equation (19)).

As we have already given the analytic solutions, we shall analyse solutions (15) and (16) numerically as examples in the next section.

3. Numerical examples

In this paper, we shall not consider the obvious static solution (14) but focus our attention on the more complicated solutions (15) and (16). In other words, we shall analyse the behaviour of $y - x$ at different time instants in (15) for $\lambda > 0$ and (16) for $\lambda = 0$. We would note also that C can only take on the values 1 or -1, as stated in Au and Fung (1982a). From equation (15), as $\sqrt{\lambda}|x| \gg 1$ and $|x| \gg 1$, we see that $y \to \pm\sqrt{\lambda}\tanh[\sqrt{\lambda}(x - 4\lambda t)]$ for $C = 1$. In other words, in a space region far from the singularities $x = \pm 1/\sqrt{\lambda}$ and $x = 0$, y is a kink–antikink solution.

For the purpose of demonstration, we shall take $C = 1$ and $\lambda = 1$ for solution (15) in our analysis. To begin our analysis, we plot the variation of y with x at different times according to (15). At $t = -2.0$, a kink (or antikink for the $-y$ solution) soliton appears on the negative x axis (figure 1(*a*)). As time elapses, the soliton propagates to the right (figure 1(*b*)). At $t = 0$, the soliton seems to disappear. We shall learn later that the soliton has entered a transient domain in space and time. In space, this transient domain is bounded by $x = \pm 1/\sqrt{\lambda}$ ($=\pm 1$ in our examples). The soliton is as if cut off at $x = -1/\sqrt{\lambda}$ and part of it appears in the range $x > 1/\sqrt{\lambda}$; the dynamic contribution, having entered branch II (figure 3), is influenced remarkably by the static contribution, and part of the solution looks like a quantum mechanical tunnelling effect. After a transient time interval of the order of 2.0 in this example (described later in figure 3), the kink (or antikink) soliton appears and propagates to the right as shown in figures 1(*c*)–(*e*). Note that close to the boundary $x = \pm 1/\sqrt{\lambda}$, the amplitude of the soliton differs from $\sqrt{\lambda}$, but at large $|x|$ and $|t|$, the asymptotic value of the amplitude is $\sqrt{\lambda}$.

When ξ (given by (17)) is equal to a constant, we interpret $v = (\mathrm{d}x/\mathrm{d}t)_{\xi=\text{constant}}$ as the velocity of the kink soliton. It is then interesting to find out how this velocity varies as the soliton travels. In figure 2 we show the velocity v (normalised to 4λ)–x plot. On reaching the boundary ($x = \pm 1/\sqrt{\lambda}$) from the left side ($x = -1$), the velocity is zero. In this case, from (11) and (13), we see that the solution $u(x, t) = -1/x^2$ decides the velocity variation since ξ is deduced from (13). It is exactly because of this that the velocity tends to $-\infty$ as x tends to zero. We observe clearly that, asymptotically, $v \to 4\lambda$ as $x \to \pm\infty$.

We now turn our attention to the interesting transient domain. We show in figures 3(*a*)–(*m*) a series of y–x graphs for the following time instants: -0.25, -0.20, -0.15,

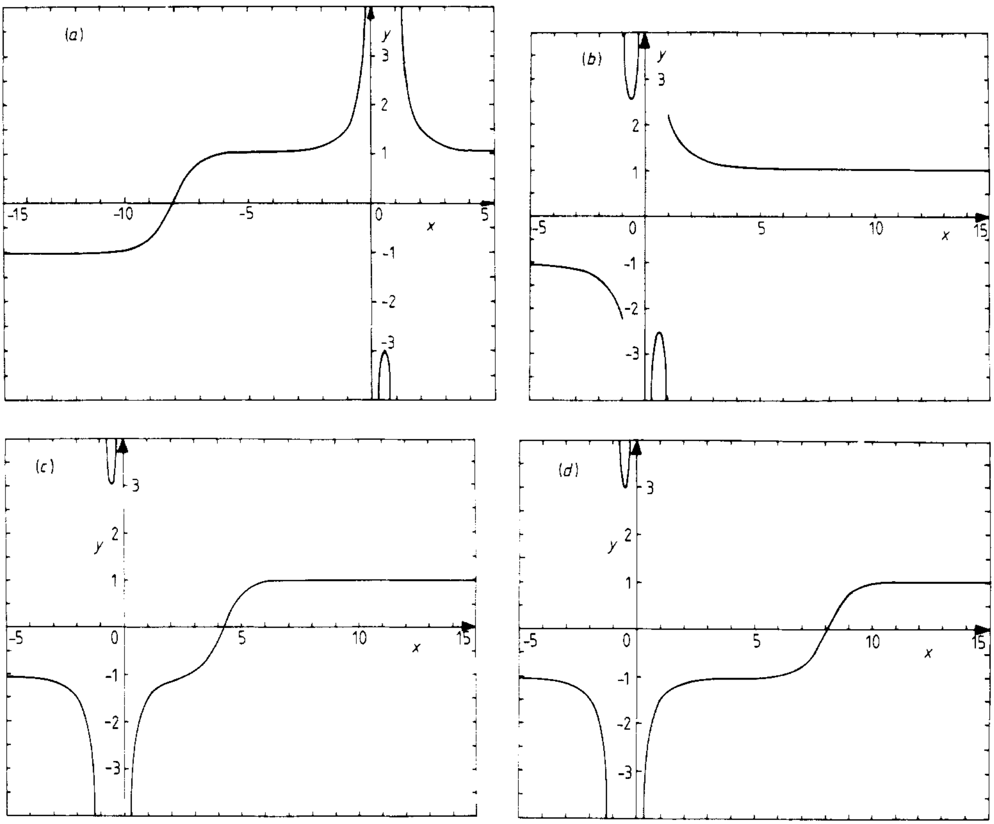

Figure 1. A kink solution propagates to the right according to solution (15) for $C = 1.0$ and $\lambda = 1.0$, and at different time instants: (*a*) $t = -2.0$, (*b*) $t = 0$, (*c*) $t = 1.0$, (*d*) $t = 2.0$.

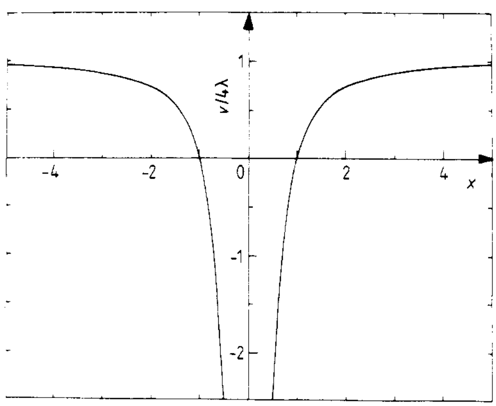

Figure 2. Dependence of the normalised velocity $v/(4\lambda)$ with x for solution (15).

−0.10, −0.05, 0, 0.05, 0.10, 0.15, 0.20, 0.25, 0.50 and 0.87. In order to see the relative contributions from the static and dynamic contributions, we show separately in each figure the static contribution (broken curve, first term of solution (15)) and

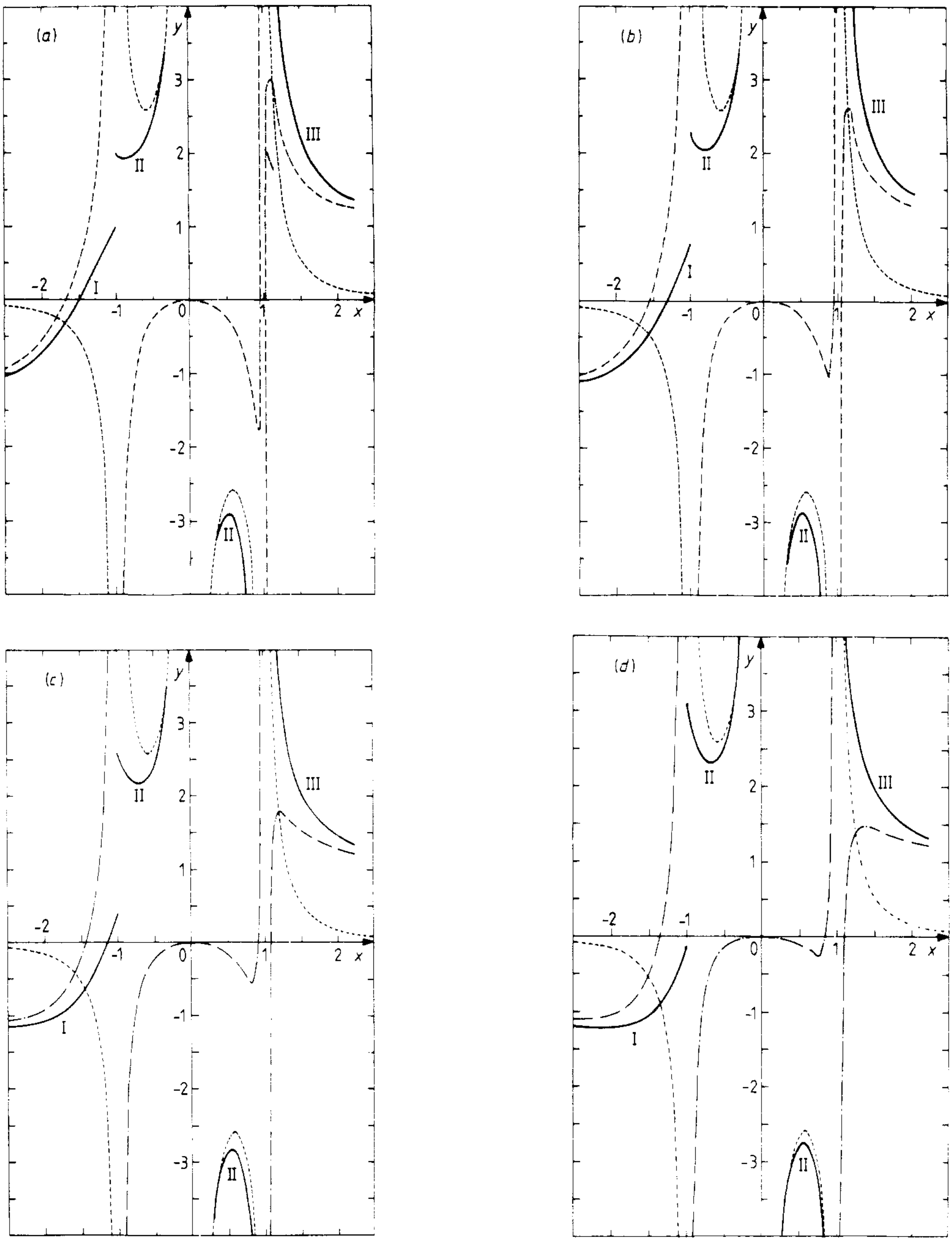

Figure 3.

the dynamic contribution (chain curve, second term of (15)). The full curve indicates the total contribution. We may separate the soliton solutions into three branches I, II and III in space, specified by

I $\qquad (-\infty < x < -1/\sqrt{\lambda})$

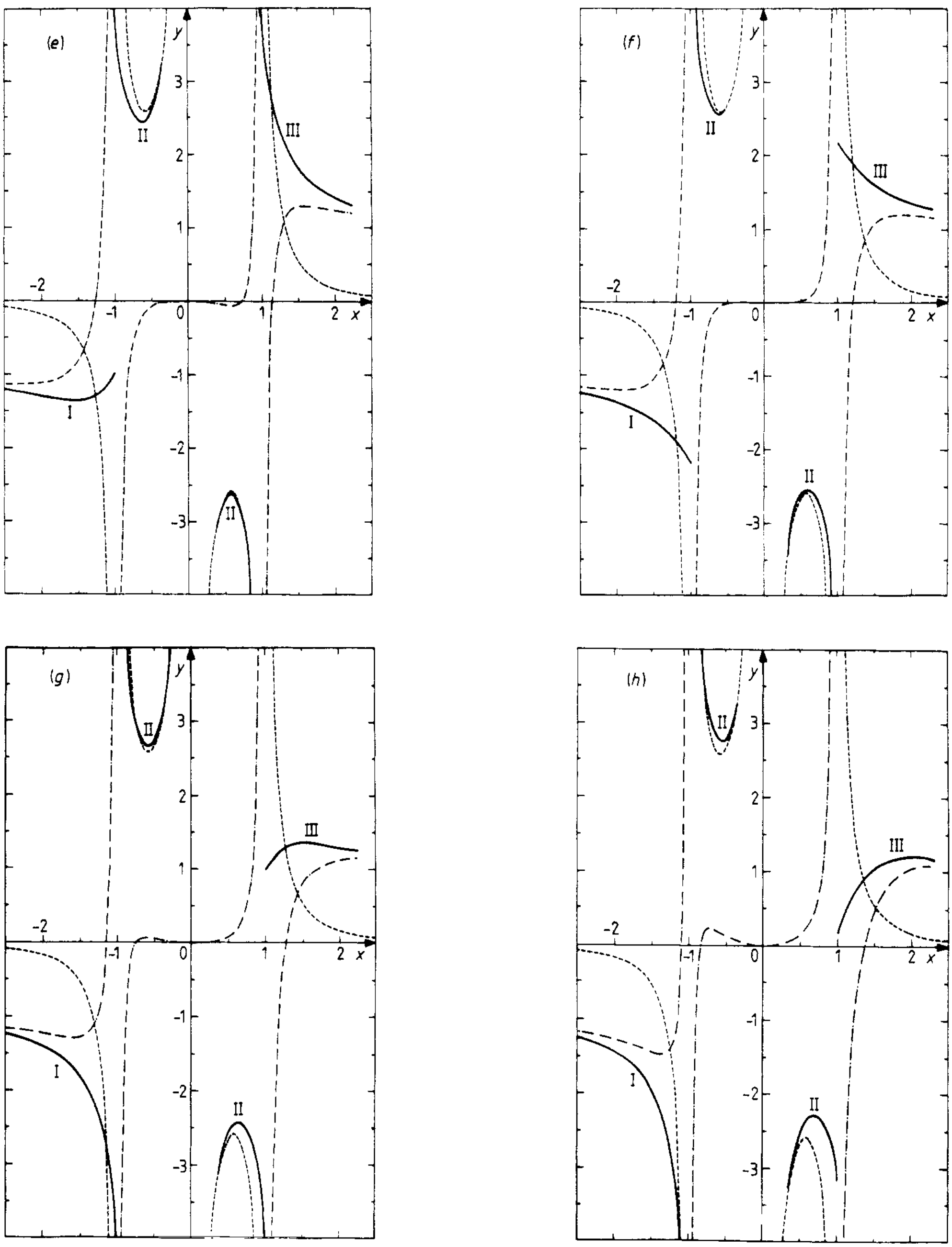

Figure 3.

II $(-1/\sqrt{\lambda} \leqslant x \leqslant 1/\sqrt{\lambda})$

III $(1/\sqrt{\lambda} < x < \infty)$.

For $b = 0$, the soliton travels from I to III.

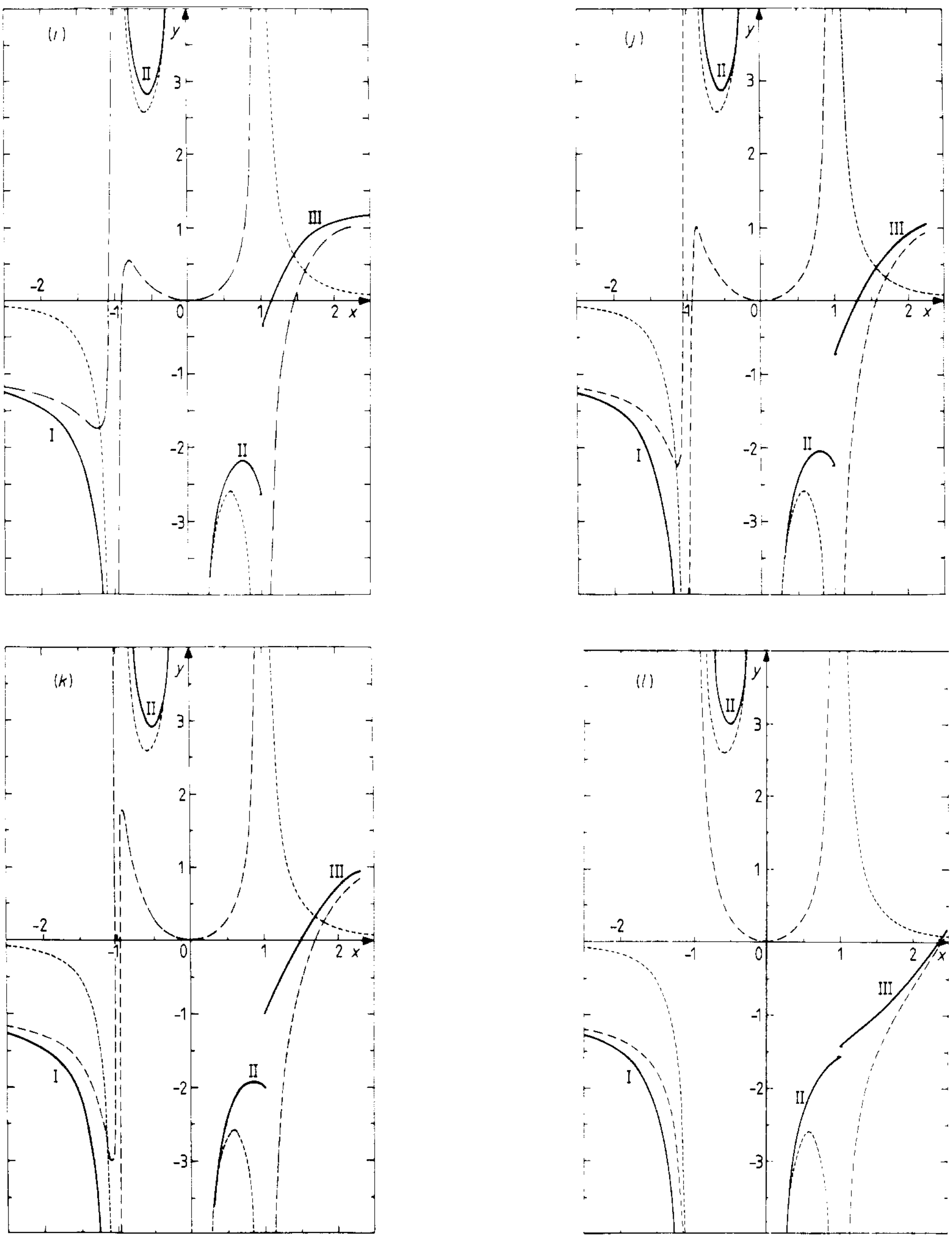

Figure 3.

As we follow the time sequence of events, we observe that the solution (rather than the soliton because it does not have the appearance of a soliton) vibrates up and down in the three branches while travelling to the right within the transient time roughly specified by $-1.0 < t < 1.0$. The two vertical lines specified by $x = \pm 1/\sqrt{\lambda} = \pm 1$, which divide the x axis into three branches, represent singularities at certain

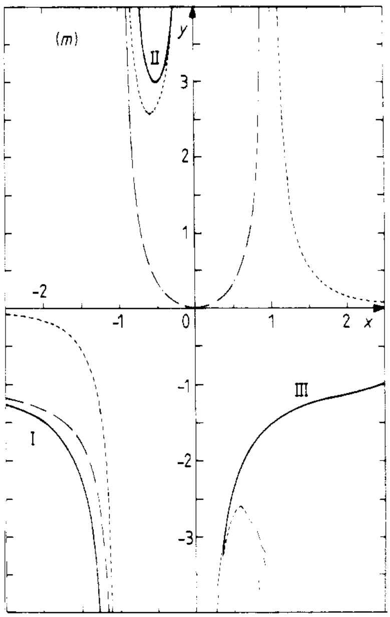

Figure 3. Transient behaviour of the kink solution (15) for $C = 1.0$, $\lambda = 1.0$ at various instants of time: (a) $t = -0.25$, (b) $t = -0.20$, (c) $t = -0.15$, (d) $t = -0.10$, (e) $t = -0.05$, (f) $t = 0$, (g) $t = 0.05$, (h) $t = 0.10$, (i) $t = 0.15$, (j) $t = 0.20$, (k) $t = 0.25$, (l) $t = 0.50$, (m) $t = 0.87$.

times. There are instants, however, when there are discontinuities of the y solutions across $x = \pm 1$, but y does not tend to positive or negative infinity. As the soliton propagates, the dynamic contribution influences the static contribution and the behaviour of y in branch II is rather complex. At the end of the transient period, branch II joins branch III and this occurs in our example slightly later than $t = 0.87$ (figure 3(m)). After that, the soliton begins its 'normal' propagation to the right, in the manner shown in figure 1.

We shall leave solution (15) and turn to solution (16) to the MKdV equation.

In figures 4(a)–(g), we show the y–x graphs according to solution (16) for $D = 0$, $t = -100, -10, 1, 0, 1, 10, 100$. The solution appears in three branches in space, but in a manner different from figure 3 ($\lambda > 0$), in the sense that there is no fixed boundary in space. At the remote past (as represented by $t = -100$), the middle branch has an inverted U shape. As time evolves, the other two branches move towards the origin,

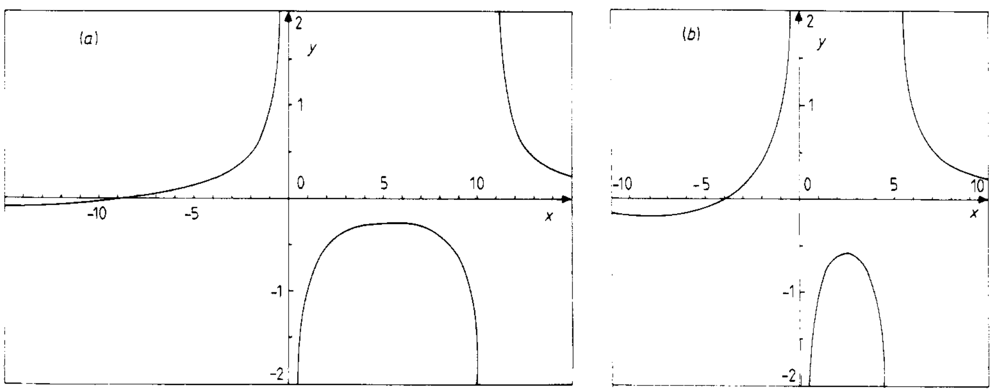

Figure 4.

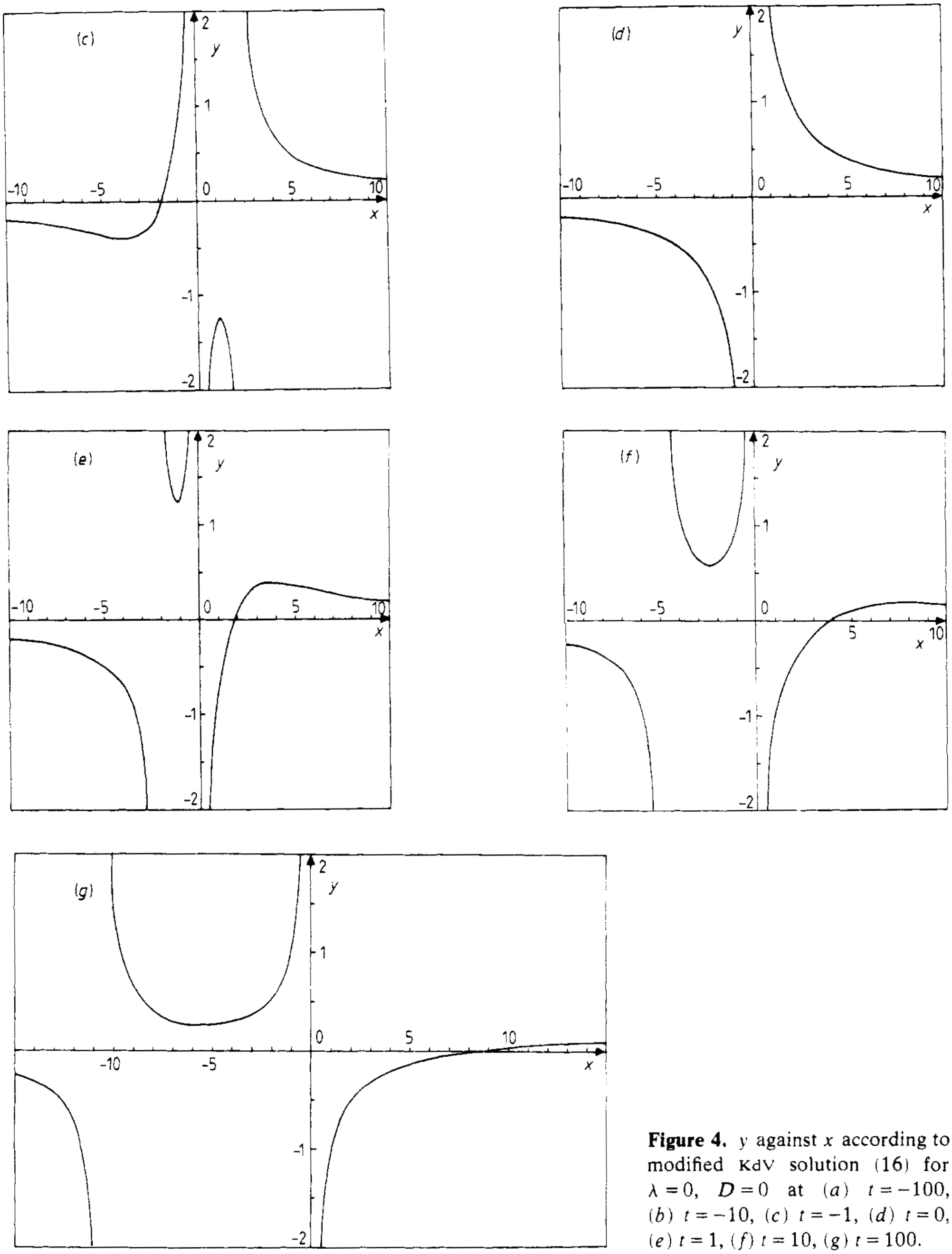

Figure 4. y against x according to modified KdV solution (16) for $\lambda=0$, $D=0$ at (*a*) $t=-100$, (*b*) $t=-10$, (*c*) $t=-1$, (*d*) $t=0$, (*e*) $t=1$, (*f*) $t=10$, (*g*) $t=100$.

and the inverted U curve is being squashed up, as shown in the development of figures 4(*a*), (*b*) and (*c*). At $t=0$ (figure 4(*e*)) the two curves become simply asymmetric with respect to the y and x axes. The curves have some type of mirror property: the figure at $t=-t_1$ (e.g. figure 4(*c*), $t_1=1$) is identical to that at $t=t_1$ (figure 4(*e*)) if both x and y are inverted. The transient behaviour of solution (16) around $t=0$ is not so complicated as indicated in figure 3 ($\lambda>0$) and we shall not present our analysis here.

4. Conclusion

In this study we have used the KdV solution $u=-1/x^2$ obtained earlier to arrive at new sets of solutions to the nonlinear equations (10) and (19) according to the theorems we have just derived. We have emphasised in § 2 that our Bäcklund transformation (5), (6) contains a generally non-zero parameter λ. Therefore we may obtain more information in solving equation (10), rather than (19), even though these two equations can be transformed to each other via (20). Such theorems are very powerful in obtaining new solutions to a nonlinear equation based on known ones of another nonlinear equation. We anticipate that our method will be very useful in bridging various nonlinear equations and deserves further developments.

We can separate the categories of our solutions into three kinds, $\lambda>0$, $\lambda=0$ and $\lambda<0$. As examples, we have analysed in detail solutions (15) for $\lambda>0$ and (16) for $\lambda=0$, the MKdV solution. Solution (15) is a set of kink–antikink soliton solutions, propagating to the right with varying velocity. The soliton passes through a transient domain in space (bounded by $x=\pm 1/\sqrt{\lambda}$) and time.

In our analytical deduction, we have studied only the simplest case where the vacuum parameter $b=0$. For non-zero b, we anticipate that solitons propagating in both left and right directions could appear. Further analysis is needed to arrive at solutions for $b\neq 0$.

Using our theorems (Fung and Au 1982), we can also obtain new KdV solutions using our solutions (14)–(16) to equation (4) presented here. In a certain sense (like velocity variation) such a KdV one-soliton solution is similar to our kink–antikink solution (15). This research will be published elsewhere (Au and Fung 1982b).

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